EMBEDDING BANACH SPACES WITH UNCONDITIONAL BASES INTO SPACES WITH SYMMETRIC BASES

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ABSTRACT

Every reflexive Banach space with unconditional basis is isomorphic to a complemented subspace of a reflexive Banach space with symmetric basis.

In this note we prove the theorem stated in the abstract. It is an improvement on a recent result of Lindenstrauss [4] who proved that every space with unconditional basis is isomorphic to a complemented subspace of a space with symmetric basis.

It will be more convenient for us to speak about sequence spaces rather than about spaces with unconditional bases.

DEFINITION. Let Z be the set of all real-valued sequences. A function $\alpha: Z \to \langle 0, \infty \rangle$ will be called a *u-norm* (*u* from "unconditional") if

(A1) $\alpha(\Sigma f_n) \leqq \Sigma\, \alpha(f_n)$, $\alpha(tf) = |t|\,\alpha(f)$ for $t \in R$

(A2) if $\alpha(f_1) < \infty$ and $f_n \downarrow 0$ (pointwise) then $\alpha(f_n) \downarrow 0$

(A3) $\alpha(e_n) > 0$ where $e_n(m) = \delta_{n,m}$

(A4) $\alpha(f) = \alpha(|f|)$.

The *sequence space* Z_α will now be defined as the set

$$Z_\alpha = \{f \in Z : \alpha(f) < \infty\}$$

equipped with the norm α.

Z_α (or α) will be called *symmetric* if

(A5) $\alpha(f) = \alpha(f \circ \pi)$ for any permutation π of natural numbers.

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Of course, sequence spaces are nothing but spaces with unconditional bases; symmetric sequence spaces are nothing but spaces with symmetric bases.

THEOREM. *For every sequence space* Z_ξ *there exists a symmetric sequence space* $X_\xi = Z_{\| \|_\xi}$ *such that* Z_ξ *is isomorphic to a complemented subspace of* X_ξ. *Moreover, reflexivity of* Z_ξ *implies reflexivity of* X_ξ.

We are going to construct the norm $\| \ \|_\xi$ in the following way:

Let (α_n) be a sequence of symmetric u-norms on Z. For every $f \in Z$ we define $Af \in Z$ by the formula

$$Af(n) = \alpha_n(f)$$

(note that $A: Z \rightarrow$ is nonlinear!).

Now, given any norm ξ we put

$$\|f\|_\xi = \xi(Af) \qquad \text{for } f \in Z.$$

It is easy to see that $\| \ \|_\xi$ is a symmetric norm.

We shall need the following two trivial lemmas. The second one is well known (cf., e.g., [3, p. 118, Lemma 4]). From now on, we assume for the sake of convenience that

$$\xi(e_n) = 1 \qquad \text{for } n = 1, 2, \cdots$$

LEMMA 1. *Let* $f_1, f_2, \cdots \in Z$ *as well as* $F_1, F_2, \cdots \in Z$ *have mutually disjoint supports. Suppose that*

(1) $\xi(F_i) \geqq 1$ *and* $F_i \leqq Af_i$ *for* $i = 1, 2, \cdots$

(2) $\sum \xi(Af_i - F_i) < K$.

Then (f_i) *regarded as a sequence in* X_ξ *is equivalent to* (F_i) *regarded as* c *sequence in* Z_ξ; *precisely*

(3) $\xi(\sum a_i F_i) \leqq \| \sum a_i f_i \|_\xi \leqq (K+1)\xi(\sum a_i F_i)$

for every sequence (a_i).

PROOF. Since both norms in (3) satisfy (A4), and (f_i) as well as (F_i) have mutually disjoint supports, it is enough to assume that $a_i \geqq 0$, $i = 1, 2, \cdots$

Denote by $i(n)$ the index such that $n \in \operatorname{supp} F_{i(n)}$ (we put $i(n) = 1$ if $n \notin \bigcup \operatorname{supp} F_i$). We have for every n

$$\alpha_n(\sum a_i f_i) \leqq \sum a_i \alpha_n(f_i) = (\sum a_i F_i)(n) + (\sum a_i(Af_i - F_i))(n)$$

$$\alpha_n(\sum a_i f_i) \geqq \alpha_n(a_{i(n)} f_{i(n)}) \geqq a_{i(n)} F_{i(n)}(n) = (\sum a_i F_i)(n).$$

Put $F = A(\Sigma\, a_i f_i)$, $G = \Sigma\, a_i F_i$ and $H = \Sigma\, a_i(Af_i - F_i)$. The above inequalities may be written as follows

$$G(n) \leqq F(n) \leqq G(n) + H(n) \qquad \text{for all } n.$$

Thus, since $\| \Sigma\, a_i f_i \|_\xi = \xi(F)$ and since ξ is monotonic, we have

$$\xi(\Sigma\, a_i F_i) \leqq \| \Sigma\, a_i f_i \|_\xi \leqq \xi(\Sigma\, a_i F_i) + \xi(H).$$

By (1) and (2),

$$\xi(H) \leqq K \max |a_i| \leqq K \xi(\Sigma\, a_i F_i)$$

which proves (3).

LEMMA 2. *If $E_1, E_2, \cdots$ are mutually disjoint, then for every symmetric norm α, the space, span $(\mathbf{1}_{E_n})$,* is a complemented subspace of Z_α ($\mathbf{1}_E$ denotes the indicator function of E).

Our construction will be based on the following

PROPOSITION.

(a) *If the norms $\gamma_n = \max_{1 \leqq i \leqq n} \alpha_i$ are reflexive* (i.e. Z_{γ_n} *are reflexive) and Z_ξ is reflexive, then $X_\xi = Z_{\| \, \|_\xi}$ is reflexive.*

(b) *If there exists a sequence $(f_i) \subset X_\xi$ of elements with mutually disjoint supports such that*

(4) $\alpha_i(f_i) = 1$

(5) $\sum_{j \neq i} \alpha_j(f_i) < 2^{-i}$

then the formula

$$I(a) = \Sigma\, a_i f_i \qquad \text{for} \quad a = (a_i) \in Z_\xi$$

defines an isomorphic embedding $I: Z_\xi \to X_\xi$.

(c) *if, in addition, the above f_i are of the form*

$$f_i(n) = \begin{cases} x_i & \text{if } k_{i-1} < n \leqq k_i \\ 0 & \text{otherwise,} \end{cases}$$

then $I(Z_\xi)$ is a complemented subspace of X_ξ.

PROOF. In (b) we have $Af_i = (\alpha_j(f_i))_{j=1}^\infty$. Take $F_i = e_i$. Then (1) and (2) in lemma 1 are clearly satisfied and thus

$$\xi(a) \leqq \| I(a) \|_\xi \leqq 2\xi(a).$$

Since $I(Z_\xi) = \operatorname{span}(f_i)$, (c) follows from Lemma 2.

To prove reflexivity of X_ξ it is enough (cf. [1, p. 76, Th. 4)] to show that if $(z_i) \subset X_\xi$ have disjoint supports, then (z_i) is not equivalent to the unit vector basis in either l_1 or c_0. Assume to the contrary that a sequence $(z_i) \subset X_\xi$ (with disjoint supports) is equivalent to the unit vector basis in either l_1 or c_0.

Denote $D_K = \operatorname{span}(z_i)_{i=K+1}^{\infty}$ (of course for every K, D_K is isomorphic to either l_1 or c_0 and hence is nonreflexive). We consider two cases:

Case 1. The following condition is satisfied

(*) $\forall_K \forall_n \forall_{\varepsilon>0} \quad \exists u \in D_K$ such that

$$\alpha_i(u) < \varepsilon \| u \|_\xi \quad \text{for } i = 1, \cdots, n.$$

This enables us to construct sequences (g_i), $(G_i) \subset Z$ so that

(**) $\xi(G_i) \geqq 1$; $G_i \leqq Ag_i$ and $\xi(Ag_i - G_i) < 2^{-i}$ for $i = 1, 2, \cdots$

(***) g_i are disjoint blocks of (z_j), i.e.

$$g_i = \sum_{j=n_i+1}^{n_{i+1}} a_j z_j \qquad \text{for some } (a_j) \text{ and } n_1 < n_2 < \cdots$$

Of course (**) and Lemma 1 imply that the space, $\operatorname{span}(g_i)$, is isomorphic to a subspace of Z_ξ and hence is reflexive, while (***) implies that $\operatorname{span}(g_i)$ is isomorphic to either c_0 or l_1 and hence is not reflexive, a contradiction.

We find (g_i), (G_i) by an easy induction. Suppose that $g_1, \cdots, g_{q-1}$ and also $G_1, \cdots, G_{q-1}$, with mutually disjoint finite supports, are already constructed so that (**) and (***) are fulfilled for $i = 1, \cdots, q-1$.

Let m be an integer such that the interval $\langle 1, m \rangle$ contains supports of $G_1, G_2, \cdots, G_{q-1}$. By (*), there exists $u \in D_n$ such that $\| u \|_\xi = 2$ and

$$\alpha_i(u) < 2^{-q-1} \cdot m^{-1} \qquad \text{for} \quad i = 1, \cdots, m.$$

Obviously we may assume that u is a block of (z_j). Also, since $u \in X_\xi$, there exists a number p such that

$$\xi(U) < 2^{-q-1} \text{ where } U = (\underbrace{0, \cdots, 0}_{p \text{ times}}, \alpha_{p+1}(u), \alpha_{p+2}(u), \cdots).$$

Now, take

$$g_q = u, \; G_q = (\underbrace{0, \cdots, 0}_{m \text{ times}}, \alpha_{m+1}(u), \cdots, \alpha_p(u), 0, 0, \cdots).$$

We have

$$\xi(Ag_q - G_q) \leqq \sum_{i=1}^{m} \alpha_i(u) + \xi(U) < 2^{-q}.$$

Thus (**) holds. This proves that Case 1 is impossible.

Case 2. Condition (*) is not satisfied. Then there exist $K, n, \varepsilon > 0$ such that for every $u \in D_K$ we have

$$\gamma_n(u) = \max_{1 \leqq i \leqq n} \alpha_i(u) \geqq \varepsilon \| u \|_\xi .$$

On the other hand, we always have

$$\gamma_n(u) \leqq \| u \|_\xi$$

and therefore on D_K, the norms γ_n and $\| \; \|_\xi$ are equivalent. However, the first is reflexive while the second is not, a contradiction.

This proves the proposition.

It now remains to construct norms (α_i) and a sequence $(f_i) \subset Z$ so that the assumptions in (a), (b), (c) are satisfied. To have (a) satisfied, α_i will be Orlicz sequence norms (for a definition, see [2]) each of them equivalent to l_2 norm.

We shall need the following trivial

LEMMA. 3. *For every $t > 0$ and every $K > 1$ there exist an Orlicz function M and numbers s, x, $0 < s < x < t$, such that*

(6) $K^{-1}y^2 \leqq M(y) \leqq Ky^2$ *for* $0 \leqq y < \infty$

$M(y) = K^{-1}y^2$ *for* $y \geqq t$ *and* $y \leqq s$

(7) $M(x) = Kx^2$.

PROOF. Let P', P'' be the half-parabolas $\{(y, K^{-1}y^2) : y \geqq 0\}$, $\{(y, Ky^2) : y \geqq 0\}$ respectively.

Let L be the straight line tangent to P'', passing through $(t, K^{-1}t^2) \in P'$, that touches P'' at the left from $(t, K^{-1}t^2)$, i.e., $L \cap P'' = \{(x, Kx^2)\}$ with $x < t$. Of course L intersects P' in one more point, say $(s, K^{-1}s^2)$. Certainly $0 < s < x < t$. Now, the diagram of M coincides with P' for $y \geqq t$ and $y \leqq s$ and it coincides with L for $s \leqq y \leqq t$. Clearly M satisfies (6) and (7).

The norms α_i will be determined by Orlicz functions M_i; these functions will be constructed as in Lemma 3. In (c) denote $m_i = k_i - k_{i-1}$. Suppose that M_i, and corresponding numbers K_i, t_i, x_i, s_i and m_i, are already defined for $i=1,\cdots,n$. We assume also that

(8) $1 = t_1 > x_1 > s_1 > t_2 > x_2 > \cdots > s_n$

$1 < K_1 < K_2 < \cdots < K_n$

$m_1 < m_2 < m_3 < \cdots < m_n$

(9) $\sum_{j \neq i\ j \leqq n} \alpha_j(f_i) < 2^{-i}$ for $i = 1,\cdots,n$

(10) $\alpha_i(f_i) = 1$ for $i = 1,\cdots,n$.

Put $N = (\inf_{i \leqq n}(2^{-i} - \sum_{j \neq i, j \leqq n} \alpha_j(f_i)))^{-1}$ and put $P = n \cdot 2^{n+1}$.

Let $K_{n+1} > \max(m_n \cdot N^2, P^2, K_n)$ and let $t_{n+1} \leqq P^{-1} \cdot s_n$ be such that

$$M_{n+1}(x_{n+1}) = K_{n+1}x_{n+1}^2$$

is a number of the form m_{n+1}^{-1} where m_{n+1} is an integer. We can do so, since x in Lemma 3 depends continuously on t and tends to 0 together with t.

We have for $i = 1,\cdots,n$

$$M_{n+1}(Nx_i) < M_{n+1}(N) = K_{n+1}^{-1} \cdot N^2 < m_n^{-1} \leqq m_i^{-1},$$

hence $m_i M_{n+1}(Nx_i) < 1$ and this means that

$$\alpha_{n+1}(f_i) < N^{-1} \leqq 2^{-i} - \sum_{j \neq i, j \leqq n} \alpha_j(f_i)$$

and hence

(9)′ $\sum_{j \neq i, j \leqq n+1} \alpha_j(f_i) < 2^{-i}$.

Take $j = 1,\cdots,n$. Since $x_{n+1} < t_{n+1}$, we have $Px_{n+1} < s_n$ and hence

$$M_j(Px_{n+1}) = K_j^{-1}P^2x_{n+1}^2 \leqq P^2x_{n+1}^2 < K_{n+1}x_{n+1}^2 = M_{n+1}(x_{n+1}).$$

Hence

$$m_{n+1}M_j(Px_{n+1}) < m_{n+1}M_{n+1}(x_{n+1}) = \alpha_{n+1}(f_{n+1}) = 1$$

and this means that

$$\alpha_j(f_{n+1}) < P^{-1} = n^{-1} \cdot 2^{-n-1} \quad \text{for } j = 1,\cdots,n.$$

Hence

(9)″ $\sum_{j \neq n+1, j \leqq n+1} \alpha_j(f_{n+1}) = \sum_{j \leqq n} \alpha_j(f_{n+1}) < 2^{-(n+1)}$.

Thus condition (9) is satisfied if we replace n by $n + 1$; clearly, (10) is also satisfied for $i = n + 1$. In this way ,we get that (9) and (10) are valid for all i and therefore (4) and (5) are satisfied.

REMARK. The main interest in our construction, as in Lindenstrauss' construction, stems from the fact that apparently special and regular spaces, namely reflexive spaces with symmetric bases, fail to have some natural properties. For example, they need not be, as was conjectured, uniformly convexifiable.

Another fact that may be of interest follows from our theorem. We deduce that L_p spaces for $1 < p < \infty$ (or more generally reflexive separable Orlicz function spaces) can be embedded into reflexive spaces with symmetric bases. This contrasts with a recent result of Lindenstrauss and Tzafriri [5, Th. 3] who proved that, unless $p = 2$, the latter spaces cannot be replaced by Orlicz sequence spaces.

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